Computational Physics

Topic 02 : Computational Problems Involving Probability

Lecture 01 : Review of Probability Concepts

Dr Kieran Murphy
Computing and Mathematics, SETU (Waterford).
(kieran.murphy@setu.ie)

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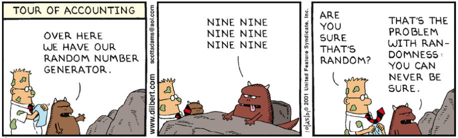
Outline

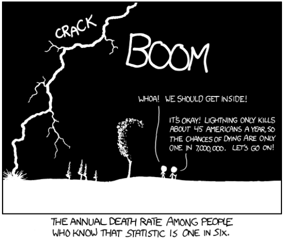
- Fundamental concepts in probability
- Laws of probability

Probability — What? Why? How?

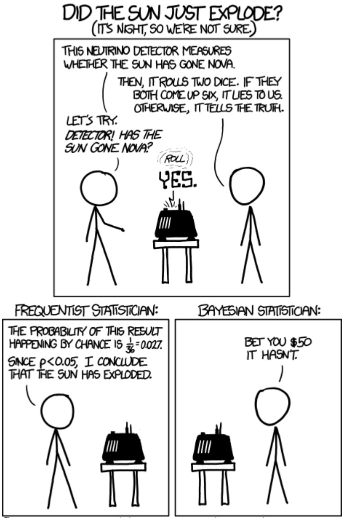


Source: http://dilbert.com/strip/2001-10-25

Source: https://xkcd.com/1152/



Source: https://xkcd.com/795/



Source: https://xkcd.com/1132/

Outline

Concept: Probability

Definition 1 (Probability)

Probability is a measure of the likelihood or chance that a particular event would occur. It is given as a number in the range 0 to 1.

human intuition only good in 0.1 to 0.9 range.

- Informal interpretation …

- Other scales can be used:
  - odds for (betting, medicine)
  - entropy (in communication theory) uses scale:

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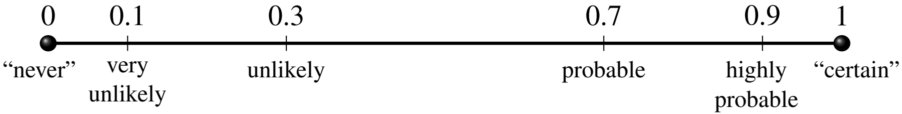

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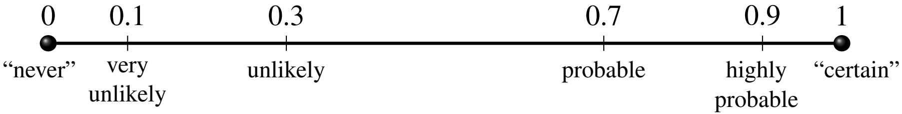

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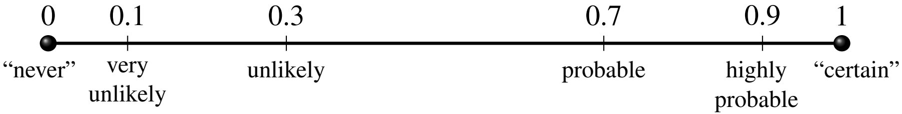

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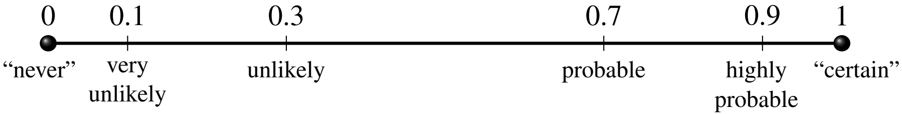

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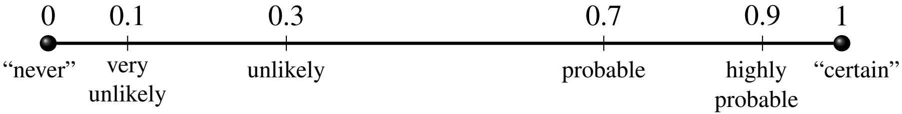

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Concept: A Probability Experiment

Definition 2 (A Probability Experiment)

A probability experiment is any process with a well-defined set of possible outcomes.

Experiment	Outcomes
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, or 6
Walk to college blindfolded	Dead, Alive

- It is important to state what the possible outcomes are in an experiment before the experiment is performed.
- The list of possible outcomes of a random experiment must be exhaustive* and mutually exclusive† At most .

*Listed outcomes covers every possibility $\Longrightarrow$ at least one
†No two listed outcomes can occur as same time $\Longrightarrow$ at most one
$\Big\}\Longrightarrow$ exactly one

Concept: Sample Space

Definition 3 (Sample Space)

Sample Space, S, is the set of all possible outcomes in the experiment,

Experiment	Sample Space
Toss a coin	$S = \{\text{Head}, \text{Tail}\}$
Roll a die	$S = \{1, 2, 3, 4, 5, 6\}$
Height of last person to join class today	$S = \{x \in \mathbb{R} \| 1 \leq x \leq 2.5\}$

- The sample space can be finite or infinite, and can be represented by categorical data, or (discrete or continuous) numerical data.
- When the sample space consists of continuous numerical data there are a few technical issues that mean we will need to tweak what we mean by having a probability of zero or a probability of one. We will worry about this later and for now in all our examples the sample space will the contain only categorical, or discrete numerical data.

Concept: Events

Definition 4 (Event)

An event is a collection (set) of some possible outcomes.

- An event may consist of a single outcome (a simple event) or consist of a number of possible outcomes (a compound event).
- The probability of any event is equal to the sum of the probabilities of the individual outcomes in the event.
- Events are sets $\Longrightarrow$ More complicated events can be constructed in terms of simpler events using set operations of intersection, union, complement (set difference between sample space and a set).
- A set of events are mutually exclusive (disjoint) if they at most one can occur.
- A set of events are exhaustive if at least one must occur.

Example — Mutually Exclusive vs Exhaustive

Example 5

- When a card is drawn from a deck at random, the four suits (hearts, diamonds, clubs and spades) are at the same time disjoint and exhaustive.
- Any event, A, and its complement, $\overline{A}$ are mutually exclusive and exhaustive pair of events.
- In the experiment of attempting this module, the events "getting a honour", or "getting a pass" are mutually exclusive but they are not exhaustive. Why?
- In my experiment of walking to college blindfolded the events "alive", "hospitalised", and "dead" are exhaustive but are not mutually exclusive. Why?

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Example 6— Set Notation Applied to Probability

Example 6

Consider the experiment of rolling a single die, and we are interested in the event of getting an even number.

- The sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

  and is represented as a set using a rectangular box. (In set theory this was called the universal set.)

- The event of interest is

$$E = \{2, 4, 6\}$$

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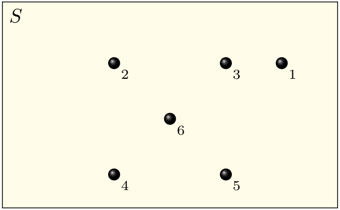

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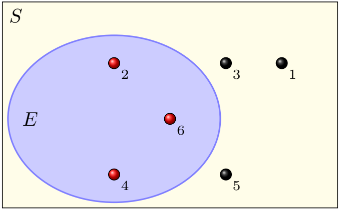

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Notation

The notation in probability theory can get a bit intense[‡], so I will try to

- Keep notation to a minimum.
- Be consistent (within this module).
- Define notation as the need arises.

Probability Law:

$$\Pr(E) = \text{“the probability that event ‘}E\text{’ occurs”} \tag{1}$$

English	Set Theory	java	python	pandas
OR	$\cup$	`||`	`or`	`|`
AND	$\cap$	`&&`	`and`	`&`
NOT	$S \setminus \bullet$ or $\bullet^c$ or $\overline{\bullet}$	!	`not`	!

[‡]For example it is not unusual to see all of the symbols x_k, x, X, $\mathcal{X}$ to represent particular aspects of the random variable x.

Outline

Assigning Probabilities

There are three main methods used in calculating probabilities. Regardless of the method used, the probability values satisfy:

1. The probability value must be in the range 0 to 1, i.e.,

Probability Law:

If an event E is a particular outcome then $\Pr(E)$ represents the probability that E will occur and

$$0 \leq \Pr(E) \leq 1 \tag{2}$$

2. The sum of the probabilities of all the possible outcomes of an experiment must equal 1, i.e.,

Probability Law: (Total Law of Probability)

If $E_1, E_2, \ldots, E_n$ are all the mutually exclusive outcomes of an experiment then

$$\Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_n) = 1 \tag{3}$$

In English — "Some outcome must happen."

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Method 1 — Classical Probability I

Classical probability theory is very much influenced by games of chance — cards, dice, roulette wheel, etc. — in such games there is a clear set of elementary outcomes which can be combined to get more complicated outcomes. Also games can be repeated ad infinitum.

Probability Law: (Classical Probability 1)

If an experiment has n possible outcomes, with all equally likely, then the probability of any one of these outcomes occurring is $1/n$.

1. Tossing a coin

$$\Pr(\text{Head}) = \Pr(\text{Tail}) = \frac{1}{2}$$

2. Rolling a die

$$\Pr(1) = \Pr(2) = \cdots = \Pr(6) = \frac{1}{6}$$

3. Picking a card from a deck of 52 cards

$$\Pr(\text{Ace of Hearts}) = \Pr(\text{Two of Hearts}) = \cdots = \frac{1}{52}$$

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Method 1 — Classical Probability II

Sometimes the result of the experiment that we want can occur a number of ways, e.g.,
Get an even number when rolling a die. Can occur 3 ways: $\{2, 4, 6\}$.

Probability Law: (Classical Probability 2))

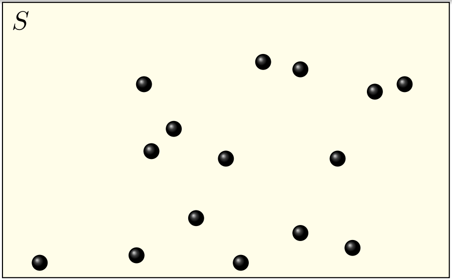
Given an experiment with each possible outcome equally likely and

S $=$ Sample Space — Set of possible outcomes
E $=$ Desired result — Set of outcomes that give the desired result

then

$$\Pr(E) = \frac{\#E}{\#S} = \frac{\text{Number of desired outcomes}}{\text{Number of possible outcomes}} \quad (4)$$

Connection to Set Notation



- The sample space, S, has size $\#S$.
  This is the number of different outcomes to the probability experiment, all of which are assumed to be equally likely.
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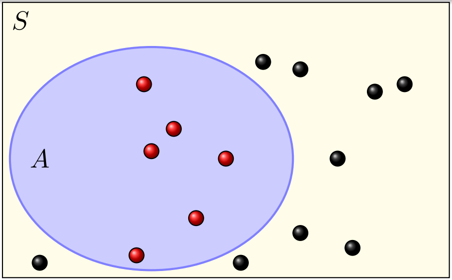

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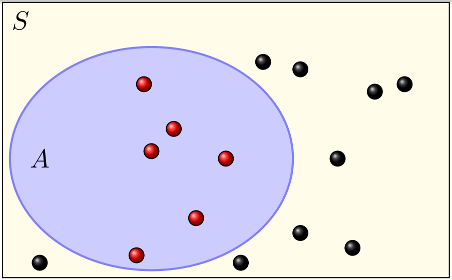

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On rolling a die what is the probability of getting:

1. an even number?
2. a number that is divisible by 3?
3. a number less than 5?

The Sample space is the same for each part, i.e.,

$$S = \{1, 2, 3, 4, 5, 6\} \qquad (\#S = 6)$$

1. *an even number ...*

$$E = \{2, 4, 6\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{3}{6} = 0.5 = 50\%.$$

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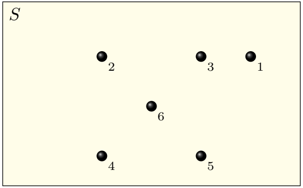

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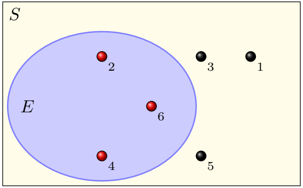
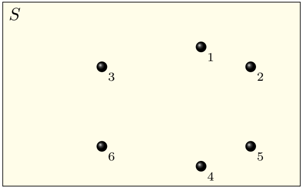


Example 7 II

2. *a number that is divisible by 3 …*

$$E = \{3, 6\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{2}{6} = 0.33333 \approx 33\%.$$



3. *a number less than 5 …*

$$E = \{1, 2, 3, 4\}$$

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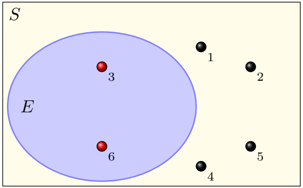

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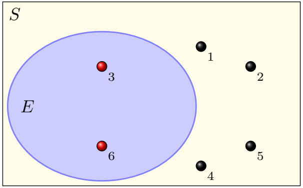

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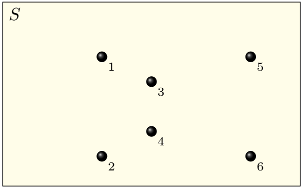

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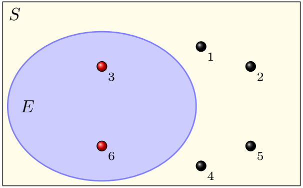

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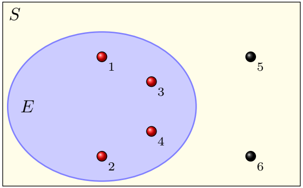


Method 2 — Frequentist Method / Historical Method I

Suppose that in a survey of 400 customers 100 were happy with the price of a particular product and 300 were unhappy with the price. The relative frequency distribution for this data is

Satisfaction	Frequency	Relative Frequency
Happy	100	0.25
Unhappy	300	0.75
	400	1.00

Using the relative frequency as probabilities we have

$$\begin{aligned} \Pr(\text{Picking a customer who is happy}) &= 0.25 \\ \Pr(\text{Picking a customer who is unhappy}) &= 0.75 \end{aligned}$$

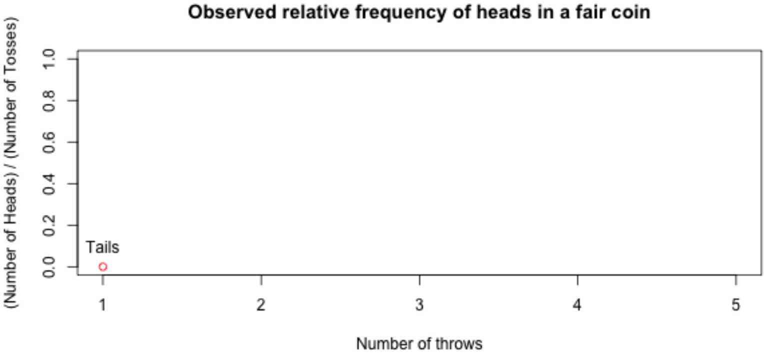
- Probabilities are assigned on the basis of experimentation or historical data.
- Classical probability would not work in this case as it is unreasonable to assume, a priori, that the two outcomes are equally likely.
- Also called the frequentist view or the empirical view of probability.

Method 2 — Frequentist Method / Historical Method II

A Frequentist assigns probability based on experience.

Definition 8 (Frequentist Probability)

Probability of a certain outcome to occur in a random experiment is the proportion of times that this outcome would occur in a very long series of repetitions of the random experiment.



Method 2 — Frequentist Method / Historical Method II

A Frequentist assigns probability based on experience.

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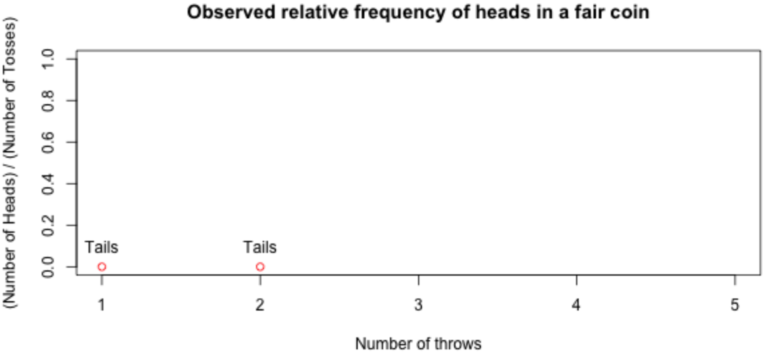

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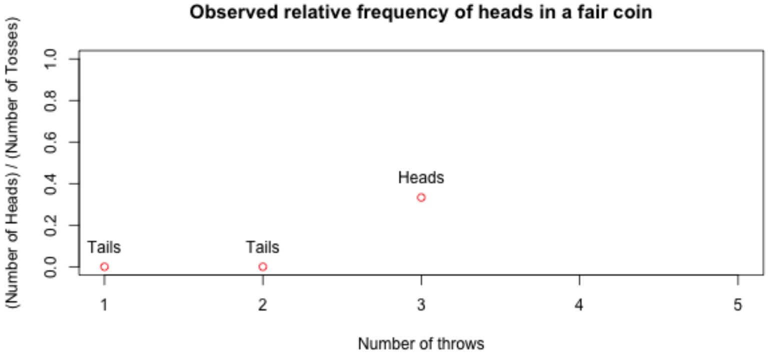

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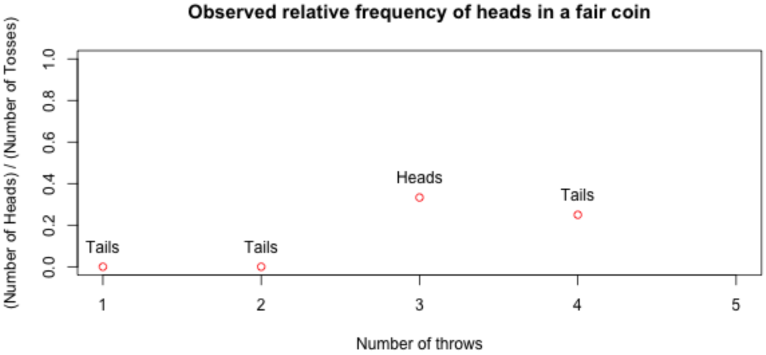

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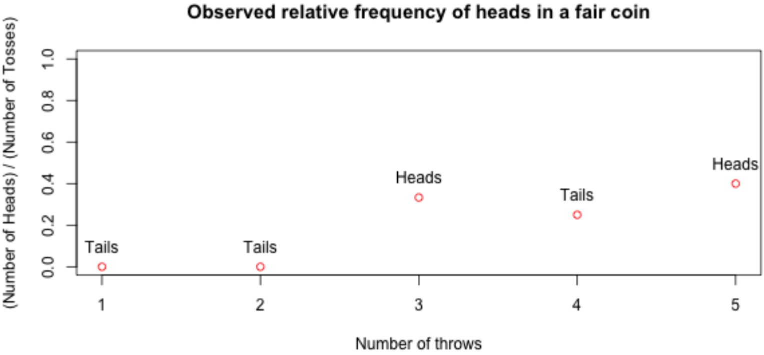

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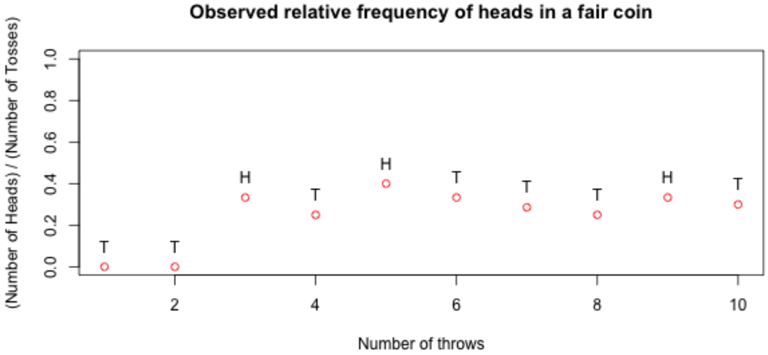

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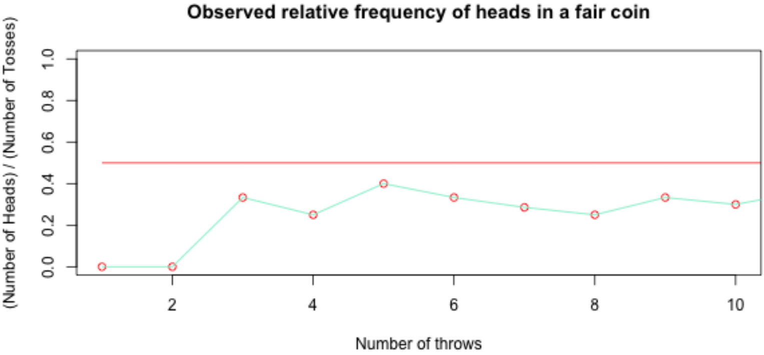

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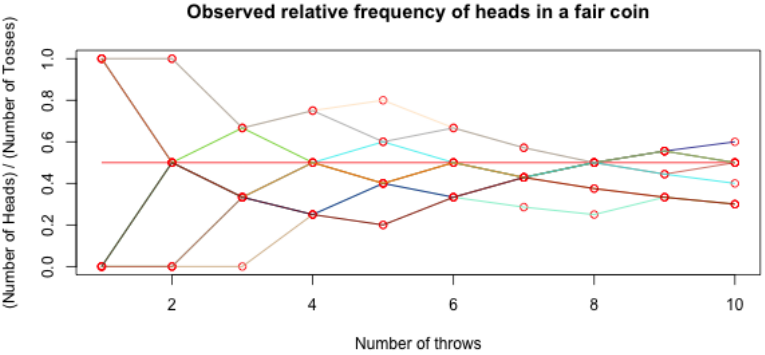

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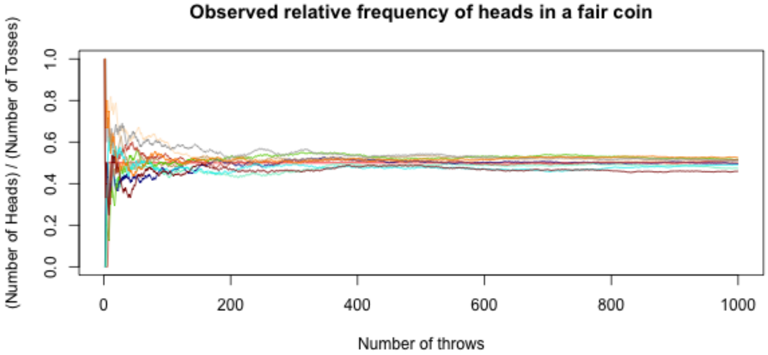

Python Implementation

1

```
import numpy as np
import numpy.random as rnd

def trial():
    coin = rnd.choice(['Heads','Tails'])
    success = coin=='Heads'
    return success

def run_trials(n):
    return sum([trial() for _ in range(n)]) / n

print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
```

```
      n        Pr(H)
=========================
      1        0.00000
     10        0.60000
    100        0.46000
   1000        0.48800
  10000        0.49660
```

Python Implementation (fast)

2

```
import numpy as np
import numpy.random as rnd

def run_trials(n):
    return rnd.choice([0,1], size=n).sum() / n

# output
print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000, 100_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
```

```
      n        Pr(H)
=========================
      1        1.00000
     10        0.80000
    100        0.45000
   1000        0.48400
  10000        0.50590
 100000        0.50174
```

For simple problems it is possible to reduce the number of calls to the random number generator by requesting multiple return values.