Computational Physics

Topic 02 : Computational Problems involving Probability

Lecture 01 : Review of Probability Concepts

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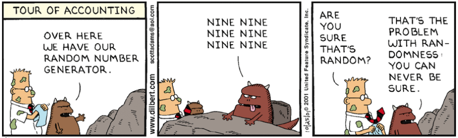
Outline

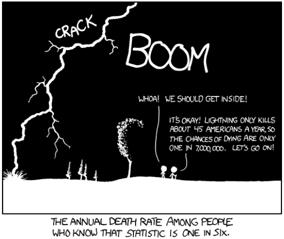
- Fundmental concepts in probability
- Laws of probability

Probability — What? Why? How?

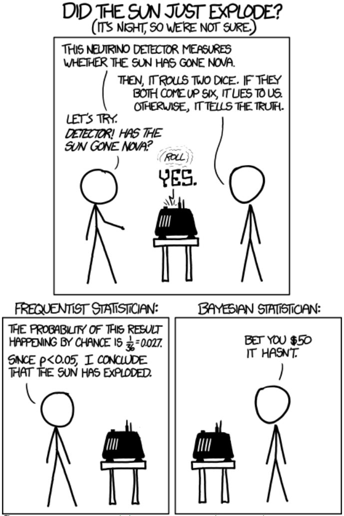


Source: http://dilbert.com/strip/2001-10-25

Source: https://xkcd.com/1152/



Source: https://xkcd.com/795/



Source: https://xkcd.com/1132/

Outline

Concept: Probability

Definition 1 (Probability)

Probability is a measure of the likelihood or chance that a particular event would occur. It is given as a number in the range 0 to 1.

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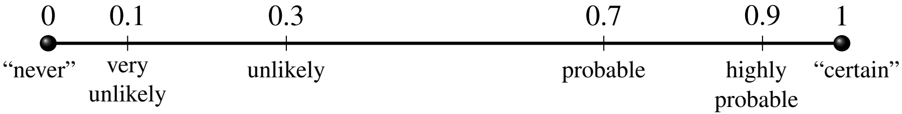

- Other scales can be used:
  - odds for (betting, medicine)

$$\text{odds for} = \frac{\text{probability}}{1 - \text{probability}} \qquad \Longleftrightarrow \qquad \text{probability} = \frac{(\text{odds for})}{(\text{odds for}) + 1}$$

  - entropy (in communication theory) uses scale $[0, \infty)$

$$\text{entropy} = (\text{probability}) \times \log_2 \left(\frac{1}{\text{probability}} \right)$$

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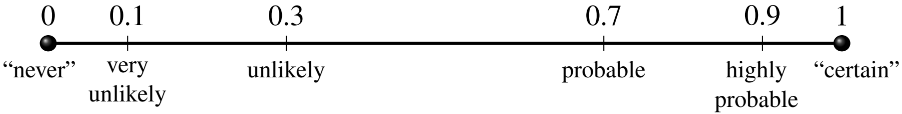

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0	0.1	0.3	0.7	0.9	1
“never”	very unlikely	unlikely	probable	highly probable	“certain”

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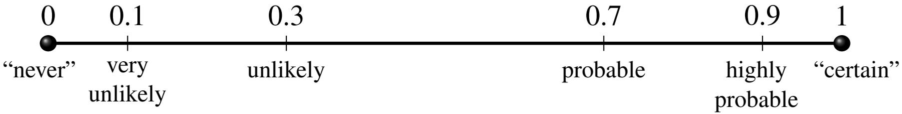

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Concept: A Probability Experiment

Definition 2 (A Probability Experiment)

A probability experiment is any process with a well-defined set of possible outcomes.

Experiment	**Outcomes**
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, or 6
Walk to college blindfolded	Dead, Alive

- It is important to state what the possible outcomes are in an experiment before the experiment is performed.
- The list of possible outcomes of a random experiment must be exhaustive* and mutually exclusive† At most .

*Listed outcomes covers every possibility $\Longrightarrow$ at least one
†No two listed outcomes can occur as same time $\Longrightarrow$ at most one
$\Big\} \Longrightarrow$ exactly one

Concept: Sample Space

Definition 3 (Sample Space)

Sample Space, S, is the set of all possible outcomes in the experiment,

Experiment	Sample Space
Toss a coin	$S = \{\text{Head}, \text{Tail}\}$
Roll a die	$S = \{1, 2, 3, 4, 5, 6\}$
Height of last person to join class today	$S = \{x \in \mathbb{R} \mid 1 \leq x \leq 2.5\}$

- The sample space can be finite or infinite, and can be represented by categorical data, or (discrete or continuous) numerical data.
- When the sample space consists of continuous numerical data there are a few technical issues that mean we will need to tweak what we mean by having a probability of zero or a probability of one. We will worry about this later and for now in all our examples the sample space will the contain only categorical, or discrete numerical data.

Concept: Events

Definition 4 (Event)

An event is a collection (set) of some possible outcomes.

- An event may consist of a single outcome (a simple event) or consist of a number of possible outcomes (a compound event).
- The probability of any event is equal to the sum of the probabilities of the individual outcomes in the event.
- Events are sets $\Longrightarrow$ More complicated events can be constructed in terms of simpler events using set operations of intersection, union, complement (set difference between sample space and a set).
- A set of events are mutually exclusive (disjoint) if they at most one can occur.
- A set of events are exhaustive if at least one must occur.

Example — Mutually Exclusive vs Exhaustive

Example 5

- When a card is drawn from a deck at random, the four suits (hearts, diamonds, clubs and spades) are at the same time disjoint and exhaustive.
- Any event, A, and its complement, $\overline{A}$ are mutually exclusive and exhaustive pair of events.
- In the experiment of attempting this module, the events “getting a honour”, or “getting a pass” are mutually exclusive but they are not exhaustive. Why?
- In my experiment of walking to college blindfolded the events “alive”, “hospitalised”, and “dead” are exhaustive but are not mutually exclusive. Why?

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Example 6— Set Notation Applied to Probability

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Consider the experiment of rolling a single die, and we are interested in the event of getting an even number.

- The sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

  and is represented as a set using a rectangular box. (In set theory this was called the universal set.)

- The event of interest is

$$E = \{2, 4, 6\}$$

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Example 6— Set Notation Applied to Probability

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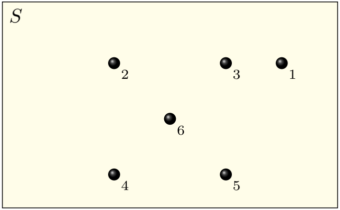

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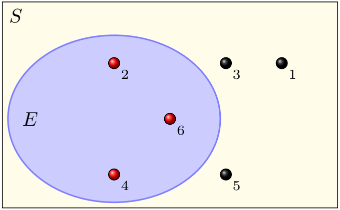

Notation

The notation in probability theory can get a bit intense[‡], so I will try to

- Keep notation to a minimum.
- Be consistent (within this module).
- Define notation as the need arises.

Probability Law:

$$\Pr(E) = \text{“the probability that event ‘}E\text{’ occurs”} \tag{1}$$

English	Set Theory	java	python	pandas
OR	$\cup$	`||`	`or`	`|`
AND	$\cap$	`&&`	`and`	`&`
NOT	$S \setminus \bullet$ or $\bullet^c$ or $\overline{\bullet}$	`!`	`not`	`!`

[‡]For example it is not unusual to see all of the symbols x_k, x, X, $\mathcal{X}$ to represent particular aspects of the random variable x.

Outline

Assigning Probabilities

There are three main methods used in calculating probabilities. Regardless of the method used, the probability values satisfy:

1. The probability value must be in the range 0 to 1, i.e.,

Probability Law:

If an event E is a particular outcome then $\Pr(E)$ represents the probability that E will occur and

$$0 \leq \Pr(E) \leq 1 \tag{2}$$

2. The sum of the probabilities of all the possible outcomes of an experiment must equal 1, i.e.,

Probability Law: (Total Law of Probability)

If $E_1, E_2, \ldots, E_n$ are all the mutually exclusive outcomes of an experiment then

$$\Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_n) = 1 \tag{3}$$

In English — "Some outcome must happen."

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Method 1 — Classical Probability I

Classical probability theory is very much influenced by games of chance — cards, dice, roulette wheel, etc. — in such games there is a clear set of elementary outcomes which can be combined to get more complicated outcomes. Also games can be repeated ad infinitum.

Probability Law: (Classical Probability 1)

If an experiment has n possible outcomes, with all equally likely, then the probability of any one of these outcomes occurring is $1/n$.

1. Tossing a coin

$$\Pr(\text{Head}) = \Pr(\text{Tail}) = \frac{1}{2}$$

2. Rolling a die

$$\Pr(1) = \Pr(2) = \cdots = \Pr(6) = \frac{1}{6}$$

3. Picking a card from a deck of 52 cards

$$\Pr(\text{Ace of Hearts}) = \Pr(\text{Two of Hearts}) = \cdots = \frac{1}{52}$$

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Method 1 — Classical Probability II

Sometimes the result of the experiment that we want can occur a number of ways, e.g.,
Get an even number when rolling a die.   Can occur 3 ways: $\{2, 4, 6\}$.

Probability Law: (Classical Probability 2))

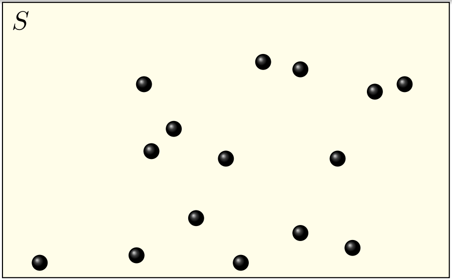
Given an experiment with each possible outcome equally likely and

S $=$ Sample Space — Set of possible outcomes
E $=$ Desired result — Set of outcomes that give the desired result

then

$$\Pr(E) = \frac{\#E}{\#S} = \frac{\text{Number of desired outcomes}}{\text{Number of possible outcomes}} \tag{4}$$

Connection to Set Notation



- The sample space, S, has size $\#S$.
  This is the number of different outcomes to the probability experiment, all of which are assumed to be equally likely.
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- The probability that event A occurs is given by

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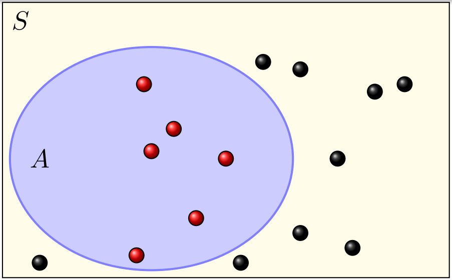

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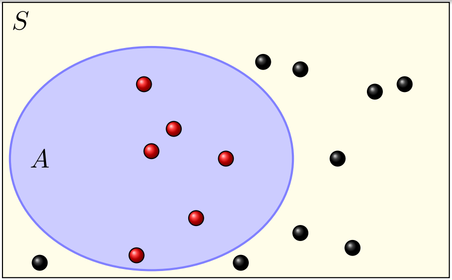

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On rolling a die what is the probability of getting:

1. an even number?
2. a number that is divisible by 3?
3. a number less than 5?

The Sample space is the same for each part, i.e.,

$$S = \{1, 2, 3, 4, 5, 6\} \qquad (\#S = 6)$$

1. *an even number …*

$$E = \{2, 4, 6\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{3}{6} = 0.5 = 50\%.$$

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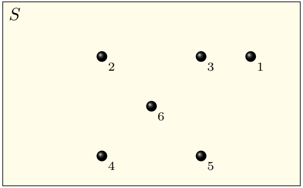

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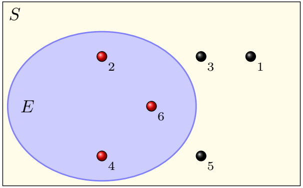
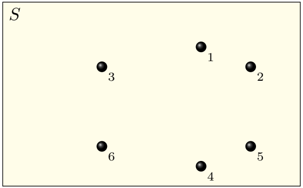


Example 7 II

2. *a number that is divisible by 3 …*

$$E = \{3, 6\}$$

$$\Pr(E) = \frac{\#E}{\#S} = \frac{2}{6} = 0.33333 \approx 33\%.$$



3. *a number less than 5 …*

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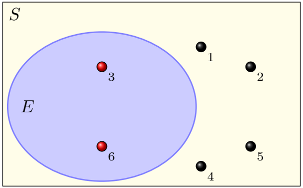

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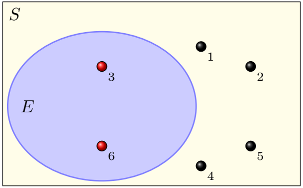

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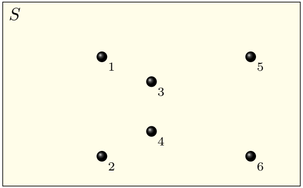

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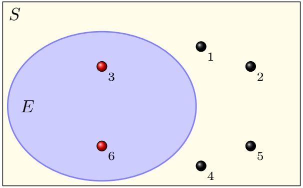

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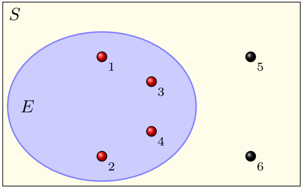


Method 2 — Frequentist Method / Historical Method   I

Suppose that in a survey of 400 customers 100 were happy with the price of a particular product and 300 were unhappy with the price. The relative frequency distribution for this data is

Satisfaction	**Frequency**	**Relative Frequency**
Happy	100	0.25
Unhappy	300	0.75
	400	1.00

Using the relative frequency as probabilities we have

$$\begin{aligned} \Pr(\text{Picking a customer who is happy}) &= 0.25 \\ \Pr(\text{Picking a customer who is unhappy}) &= 0.75 \end{aligned}$$

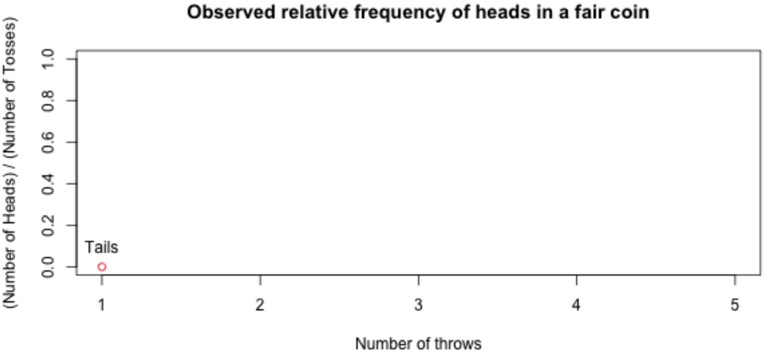
- Probabilities are assigned on the basis of experimentation or historical data.
- Classical probability would not work in this case as it is unreasonable to assume, a priori, that the two outcomes are equally likely.
- Also called the frequentist view or the empirical view of probability.

Method 2 — Frequentist Method / Historical Method II

A Frequentist assigns probability based on experience.

Definition 8 (Frequentist Probability)

Probability of a certain outcome to occur in a random experiment is the proportion of times that this outcome would occur in a very long series of repetitions of the random experiment.



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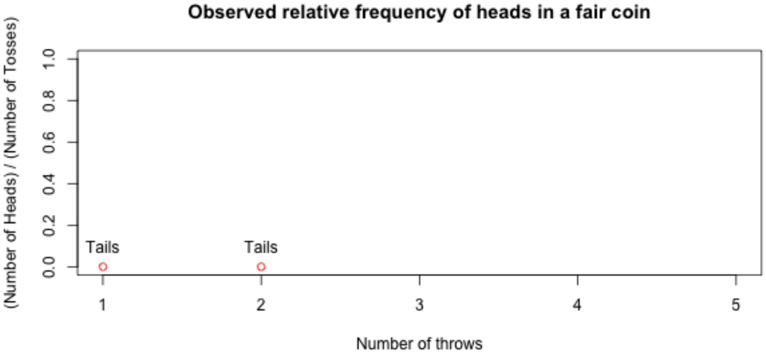
Probability of a certain outcome to occur in a random experiment is the proportion of times that this outcome would occur in a very long series of repetitions of the random experiment.



Method 2 — Frequentist Method / Historical Method II

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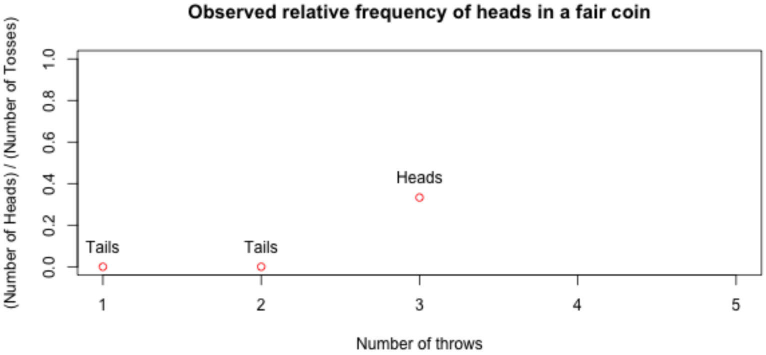

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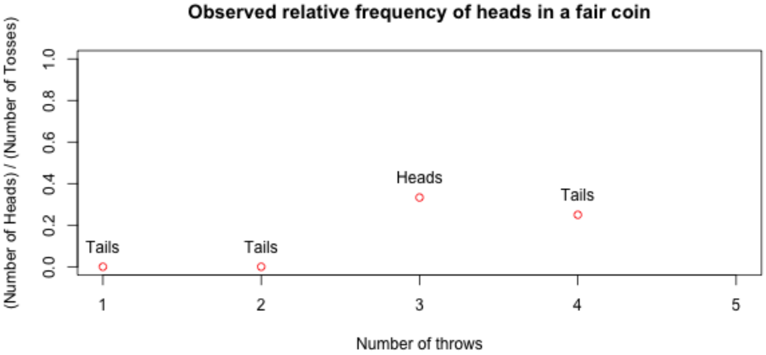

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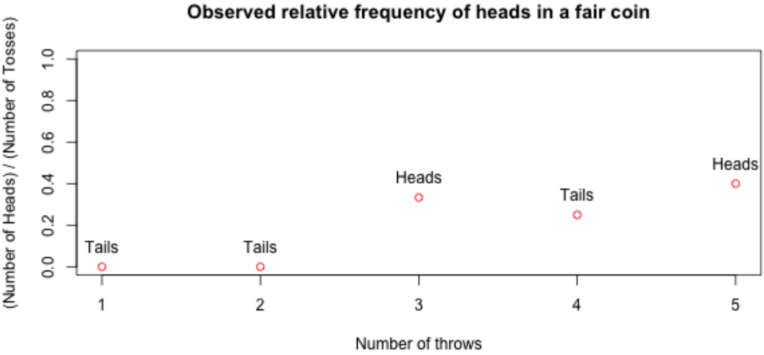

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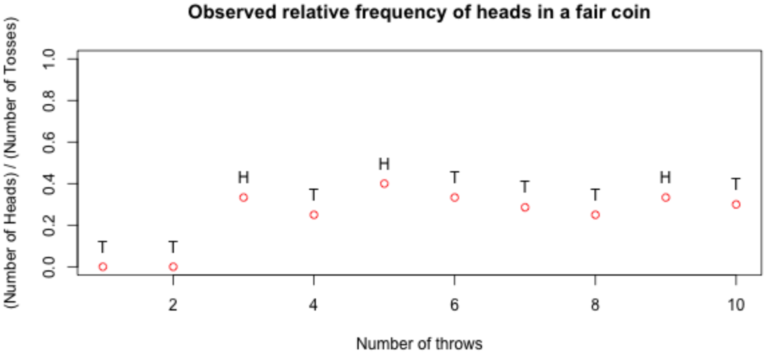

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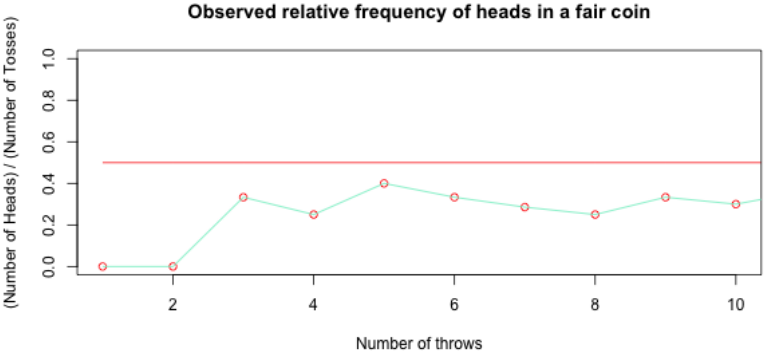

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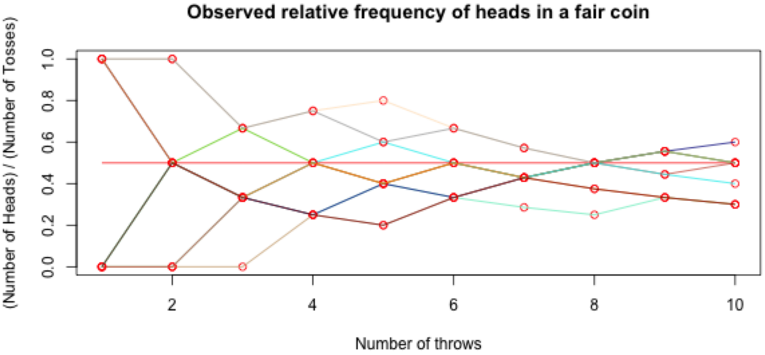

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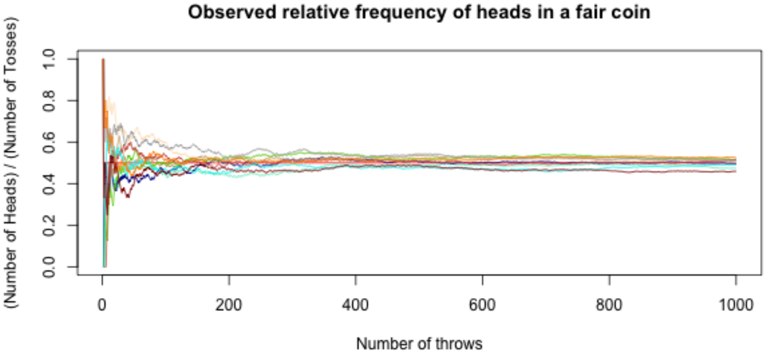

Python Implementation

```
import numpy as np
import numpy.random as rnd

def trial():
    coin = rnd.choice(['Heads','Tails'])
    success = coin=='Heads'
    return success

def run_trials(n):
    return sum([trial() for _ in range(n)]) / n

print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
```

```
      n         Pr(H)
=========================
      1         0.00000
     10         0.60000
    100         0.46000
   1000         0.48800
  10000         0.49660
```

Python Implementation (fast)

```
import numpy as np
import numpy.random as rnd

def run_trials(n):
    return rnd.choice([0,1], size=n).sum() / n

# output
print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000, 100_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
```

```
      n      Pr(H)
=========================
      1      1.00000
     10      0.80000
    100      0.45000
   1000      0.48400
  10000      0.50590
 100000      0.50174
```

For simple problems it is possible to reduce the number of calls to the random number generator by requesting multiple return values.